

Alternation of sign of magnetization current in driven XXZ chains with twisted XY boundary gradients

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We investigate an open XXZ spin $1/2$ chain driven out of equilibrium by coupling with boundary reservoirs targeting different spin orientations in XY plane. Symmetries of the model are revealed which appear to be different for spin chains of odd and even sizes. As a result, spin current is found to alternate with chain length, ruling out the possibility of ballistic transport. Heat transport is switched off completely by virtue of another global symmetry. Further, we investigate the model numerically and analytically. At strong coupling, we find exact nonequilibrium steady state using a perturbation theory. The state is determined by solving secular conditions which guarantee self-consistency of the perturbative expansion. We find nontrivial dependence of the magnetization current on the spin chain anisotropy Δ in the critical region $|\Delta| < 1$, and a phenomenon of tripling of the twisting angle along the chain for narrow lacunes of Δ .

I. INTRODUCTION

Manipulating of quantum systems which consist of a few quantum dots or quantum bits forms a basis of a functioning of any quantum computing device. Recent experimental advances allow to assemble and manipulate nanomagnets consisting of just a few atoms and perform measurements on nanostructures with an atomic resolution. However, a theoretical understanding of microscopic quantum systems out of equilibrium (e.g. under constant pumping or continuous measurement by a quantum probe) is far from being complete, apart from simplest cases like a single two-level system or a quantum harmonic oscillator under external pumping or in contact with a reservoir.

Coupling of a quantum system with an environment (or with a measuring apparatus) is described under standard assumptions [1], [2] in the framework of a Lindblad Master equation (LME) for a reduced density matrix, where a unitary evolving part is complemented with a Lindblad dissipative action. Under the LME dynamics, a system with gradients evolves towards a nonequilibrium steady state (NESS) with currents.

The Heisenberg model of interacting spins is an oldest many-body quantum model of a strongly correlated systems, important both from a theoretical and an experimental point of view. Energy and magnetization transport in the XXZ Heisenberg spin chain is extensively studied by different methods, see [3, 5] for reviews, including Bethe Ansatz [4, 6, 7], bosonization [8], [9], Lagrange multipliers [10], exact diagonalization [11, 12], Lanczos method [13], quantum Monte Carlo [14], DMRG [15, 16]. Investigation of nonequilibrium spin chains from the point of view of Markovian Master equations contributes to the "equilibrium" studies. In quasi one-dimensional spin chain materials like SrCuO_2 many transport characteristics are measurable experimentally [18].

Investigation of NESS in driven spin chains is hindered by enormous technical difficulties due to an exponentially growing Hilbert space, although time-dependent DMRG method allows to reach somewhat larger sizes [17]. In many studies the NESS are produced by coupling a spin chain at the edges to boundary baths of polarization aligning boundary spins along the anisotropy axis Z , so that the first and the last spin tend to be antiparallel [31]. For this particular choice of boundary baths, a substantial progress in understanding the properties of a driven XXZ model have been recently achieved [19–24, 26–30], including an exact solution of the problem in case of strong boundary driving [23], see also [24] and references therein. Note that within this boundary setup, one has the transport of a z -component of magnetization j^z , caused by boundary gradients along the z -axis, thus probing diagonal elements of a diffusion matrix and a conductivity tensor.

In present paper we carry out a systematic study of nonequilibrium steady states in driven XXZ spin chain resulting from boundary gradients in the XY direction, transverse to the anisotropy axis. In the context of spin and energy transport and for appropriate limits, this would correspond to probing the nondiagonal elements of the diffusion and conductivity tensors. To this end, we bring an XXZ spin chain in contact with reservoirs aligning the boundary spins along the X -axis at one end and along the Y -axis in the other end, thus trying to impose a twisting angle in XY -plane of $\pi/2$ between the first and the last spin. Within the LME formalism, twisting angles $\alpha = 0$ or $\alpha = \pi$ result in a complete suppression of the magnetization current, see a remark after Eq.(16), making a setup with $\alpha = \pi/2$ the simplest nontrivial one. We derive general symmetries of the model which indicate that magnetization current should alternate its sign with system size, for any value of boundary coupling. On this basis we rule out the possibility of a ballistic spin current, which is typically observed in integrable XXZ model under other kinds of perturbations [25],[33]. We argue, further, that the phenomenon of the spin current alternation is rather general and in particular is not related to the integrability.

To confirm further our findings we investigate the nonequilibrium steady state of the XXZ model numerically and analytically, for varying amplitudes of boundary couplings and boundary gradients. We find complete agreement with our predictions

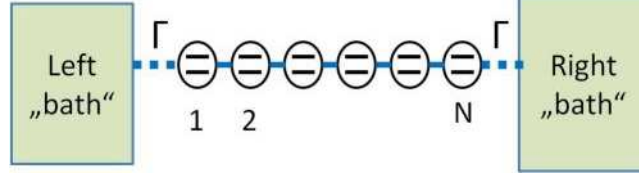


Figure 1: Schematic layout of the problem. The chain of two-level systems coupled at the boundaries to the reservoirs. Dissipation introduced by the reservoirs is described by a quantum Master equation (1)

which in particular are confirmed analytically, in the regime of strong coupling. In this regime, full analytic solution of the Lindblad equation for large Lindblad amplitudes demonstrates a very nonmonotonic dependence of the magnetization current on the system size and anisotropy.

The plan of the paper is the following. In sec.II we introduce the model. Symmetries of the Lindblad master equation are obtained in sec.III. Using the symmetries general NESS properties for arbitrary system size and coupling are revealed, including an admissibility of spin and energy currents. Parity-depending LME symmetries lead to a phenomenon of magnetization current sign alternation with the system size. In sec. IV our predictions are tested further by calculating the NESS analytically and numerically for small system sizes. The cases of XX model and of XXZ model are treated separately in sec.IV A and sec.IV B. Limits of weak coupling and of weak driving force are briefly discussed at the end of sec.IV C. Appendix contain necessary technical details. In conclusion we summarize our findings.

II. THE MODEL

We study an open chain of N quantum spins in contact with boundary reservoirs. The time evolution of the reduced density matrix ρ is described by a quantum Master equation in the Lindblad form [1],[2], [29] (here and below we set $\hbar = 1$)

$$\frac{\partial \rho}{\partial t} = -i [H, \rho] + \Gamma(\mathcal{L}_L[\rho] + \mathcal{L}_R[\rho]), \quad (1)$$

where H is the Hamiltonian of an extended quantum system, and $\mathcal{L}_L[\rho]$ and $\mathcal{L}_R[\rho]$ are Lindblad dissipators acting on spatial edges of the system. This setting, shown schematically in Fig.1, is a common starting point in studies of nonequilibrium transport, a particular choice of \mathcal{L}_L and \mathcal{L}_R depending on the application. Often, a choice of the Lindblad action is operational, i.e. is determined by a requirement to favour a relaxation of a system or its part towards a target state with given properties, e.g. to a state with given polarization or temperature [17]. In this way, one describes an effective coupling of the system, or a part of it, with a respective bath-environment. Note that within the quantum protocol of repeated interactions, [29], [34] the equation (1) appears to describe an exact time evolution provided that the coupling to reservoir is rescaled appropriately with the time interval between consecutive interactions of the system with the reservoirs.

We specify the Hamiltonian to describe an open XXZ spin chain with an anisotropy Δ

$$H = \sum_{k=1}^{N-1} (\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \Delta \sigma_k^z \sigma_{k+1}^z), \quad (2)$$

while \mathcal{L}_L and \mathcal{L}_R are Lindblad dissipators favouring a relaxation of boundary spins $k = 1$ and $k = N$ towards states described by one-site density matrices ρ_L and ρ_R , i.e. $\mathcal{L}_L[\rho_L] = 0$ and $\mathcal{L}_R[\rho_R] = 0$. General form of the matrices ρ_L and ρ_R is $\rho_{L,R} = \frac{1}{2} + \alpha_{L,R} \sigma^x + \beta_{L,R} \sigma^y + \gamma_{L,R} \sigma^z$, where σ^a are Pauli matrices, and α, β, γ are real constants satisfying $\text{Tr} \rho_{L,R}^2 \leq 1$ or $\alpha_{L,R}^2 + \beta_{L,R}^2 + \gamma_{L,R}^2 \leq 1$. For a general choice of the Lindblad dissipator $\mathcal{L}_{L,R}[\cdot] = \sum_{i,j=1}^3 a_{ij}^{L,R} (\sigma^i \cdot \sigma^j - \frac{1}{2} \{ \cdot, \sigma^j \sigma^i \})$, each of the 2×2 matrices ρ_L and ρ_R , if not constant, belongs to at most one-dimensional manifold [36].

We choose boundary reservoirs which tend to align the spin at the left edge along the x direction and the spin at the right edge along the y direction. Consequently, the reservoirs try to establish a twisting angle of $\pi/2$ between the first and the last spin in the chain. Such a setting is achieved by taking the Lindblad action in the form

$$\begin{aligned}\mathcal{L}_L[\rho] &= -\frac{1}{2} \sum_{m=1}^2 \{\rho, W_m^\dagger W_m\} + \sum_{m=1}^2 W_m \rho W_m^\dagger, \\ \mathcal{L}_R[\rho] &= -\frac{1}{2} \sum_{m=1}^2 \{\rho, V_m^\dagger V_m\} + \sum_{m=1}^2 V_m \rho V_m^\dagger.\end{aligned}\tag{3}$$

Here W_m and V_m are polarization targeting Lindblad operators, which act on the first and on the last spin respectively,

$$\begin{aligned}W_1 &= \sqrt{\frac{1-\kappa}{2}}(\sigma_1^z + i\sigma_1^x), \\ W_2 &= \sqrt{\frac{1+\kappa}{2}}(\sigma_1^z - i\sigma_1^x), \\ V_1 &= \sqrt{\frac{1+\kappa'}{2}}(\sigma_N^y + i\sigma_N^z), \\ V_2 &= \sqrt{\frac{1-\kappa'}{2}}(\sigma_N^y - i\sigma_N^z).\end{aligned}\tag{4}$$

In absence of the unitary term in (1) the boundary spins relax with a characteristic time Γ^{-1} [37] to specific states described via *one-site* density matrices ρ_L and ρ_R , satisfying $\mathcal{L}_L[\rho_L] = 0$ and $\mathcal{L}_R[\rho_R] = 0$, where

$$\rho_L = \frac{1}{2}(I - \kappa\sigma_1^y)\tag{5}$$

$$\rho_R = \frac{1}{2}(I + \kappa'\sigma_N^x).\tag{6}$$

From the definition of a one-site density matrix $\rho_{\text{one-site}} = \frac{1}{2}(I + \sum \langle \sigma^\alpha \rangle \sigma^\alpha)$, we see that the Lindblad superoperators \mathcal{L}_L and \mathcal{L}_R indeed try to impose a twisting angle of $\pi/2$ in XY plane between the first and the last spin [38]. The twisting gradient drives the system in a steady-state with currents. In the following we restrict to a symmetric choice

$$0 \leq \kappa = \kappa' \leq 1.\tag{7}$$

The parameter κ determines the amplitude of the gradient between the left and right boundary, and therefore plays a fundamentally different role from the coupling strength Γ . The limits $\kappa = 1$ and $\kappa \ll 1$ will be referred to as the strong driving and weak driving case respectively. The two limits describe very different physical situations as exemplified in other NESS studies.

The reason for choosing the Lindblad dissipators which impose a twisting gradient in XY -plane perpendicular to the Heisenberg chain anisotropy is two-fold. Firstly, our choice is "orthogonal" and therefore complementary to usually considered boundary setup [19]-[24],[26]-[28], where alignment of spins at the boundary is parallel to the anisotropy axis Z . Our system is not mappable to a quadratic fermionic system, and is therefore not amenable to a analysis via a canonical quantization in the Fock space of operators [19],[20], which can be used for usual boundary setup. Secondly, one realizes that imposing both X - and Y -boundary gradients is necessary to create a nonvanishing stationary Z -magnetization current. A boundary setup with parallel or antiparallel boundary spins, aligned in the same direction in XY - plane leads to zero steady magnetization current, see Remark after (17). Consequently, by imposing a twisting angle in XY -plane of $\pi/2$ we choose a minimal nontrivial setup.

Transport properties in the XXZ spin chain $H = \sum_n h_{n,n+1}$ are governed by spin and energy current operators which are defined by lattice continuity equations $\frac{d}{dt}\sigma_n^z = \hat{j}_{n-1,n} - \hat{j}_{n,n+1}$, $\frac{d}{dt}h_{n,n+1} = \hat{j}_n^Q - \hat{j}_{n+1}^Q$ where

$$\hat{j}_{n,m} = 2(\sigma_n^x \sigma_m^y - \sigma_n^y \sigma_m^x),\tag{8}$$

$$\hat{j}_n^Q = -\sigma_n^z \hat{j}_{n-1,n+1} + \Delta(\hat{j}_{n-1,n} \sigma_{n+1}^z + \sigma_{n-1}^z \hat{j}_{n,n+1}).\tag{9}$$

We show next that the magnetization current j alternates its sign with the system size, while the energy transport is switched off completely, if $\kappa = \kappa'$, despite the boundary gradients. Being locally conserved quantities, in the stationary state the currents $\langle \hat{j}_{n,n+1} \rangle = j$, $\langle \hat{j}_n^Q \rangle = J^Q$ are equal across all the bonds.

For the following is essential to note, that for our choice of the Lindblad dissipator (3), (4), the system (1) has a *unique* nonequilibrium steady state. Its existence and uniqueness for any coupling Γ is guaranteed by the completeness of the algebra, generated by the set of operators $\{H, V_m, W_m, V_m^+, W_m^+\}$ under multiplication and addition [39], and is verified straightforwardly as in [40].

III. SYMMETRIES OF THE LINDBLAD EQUATION AND RESTRICTIONS ON ENERGY AND MAGNETIZATION CURRENTS

Let us denote by $\rho(N, \Delta, t)$ a time-dependent solution of the Lindblad equation (1), (4) for a system of N sites, and by $\rho(N, \Delta) = \lim_{t \rightarrow \infty} \rho(N, \Delta, t)$ its steady state solution, where Δ is the anisotropy of the XXZ Hamiltonian (2). There exist transformations which map one LME solution to another. Some of the transformations depend on the parity of N . For *even* N , introducing $U = U^\dagger = \prod_{n \text{ odd}} \otimes \sigma_n^z$, we find that $\rho(N, -\Delta, t) = U \rho^*(N, \Delta, t) U$ is a solution of the same Lindblad equation with the anisotropy $-\Delta$. We denote by ρ^* a complex conjugated matrix in the basis where σ^z is diagonal. However, if a nonequilibrium steady state is unique, then a relation follows which relates the NESS for positive and negative Δ ,

$$\rho(N, -\Delta) = U \rho^*(N, \Delta) U \text{ for } N \text{ even.} \quad (10)$$

For *odd* N we obtain another relation for NESS, introducing a unitary transformation $\Sigma_y = (\sigma^y)^{\otimes N}$

$$\rho(N, -\Delta) = \Sigma_y U \rho^*(N, \Delta) U \Sigma_y \text{ for } N \text{ odd} \quad (11)$$

Another LME symmetry does not depend on parity of N and maps a NESS onto itself provided that an additional condition (7) is satisfied:

$$\rho(N, \Delta) = \Sigma_x U_{rot} R \rho(N, \Delta) R U_{rot}^\dagger \Sigma_x, \quad (12)$$

where $R(A \otimes B \otimes \dots \otimes C) = (C \otimes \dots \otimes B \otimes A)R$ is a left-right reflection, and the diagonal matrix $U_{rot} = \text{diag}(1, i)^{\otimes N}$ is a rotation in XY plane, $U_{rot} \sigma_n^x U_{rot}^\dagger = \sigma_n^y$, $U_{rot} \sigma_n^y U_{rot}^\dagger = -\sigma_n^x$ and $\Sigma_x = (\sigma^x)^{\otimes N}$. Note also that at the point $\Delta = 0$ symmetries (10) and (11) map a NESS onto itself, and are independent on (12).

The relations (10) – (12) impose restrictions on the NESS density matrices and respectively on all observables including the magnetic and energy currents $j = \text{Tr}(\rho \hat{j}_{n,n+1})$ and $J^Q = \text{Tr}(\rho \hat{J}_n^Q)$ where $\hat{j}_{n,n+1}$ and \hat{J}_n^Q are given by (8) and (9). We drop the subscripts since in the steady state the currents j, J^Q are equal across all bonds. By virtue of the symmetry (10), we obtain for N even

$$\begin{aligned} j(-\Delta) &= \text{Tr}(\hat{j}_{n,n+1} \rho(N, -\Delta)) = \\ &= \text{Tr}(\hat{j}_{n,n+1} U \rho^*(N, \Delta) U) = \\ &= -\text{Tr}(U \hat{j}_{n,n+1} U \rho(N, \Delta))^* = \\ &= \text{Tr}(\hat{j}_{n,n+1} \rho(N, \Delta)) = j(\Delta). \end{aligned}$$

In the last passage we used (i) $\hat{j}_{n,n+1}^* = -\hat{j}_{n,n+1}$ in the basis where σ^z is diagonal (ii) $U \hat{j}_{n,n+1} U = -\hat{j}_{n,n+1}$ (iii) the fact that an observable j is a real number. Analogously, for odd N , we obtain using (11), that $j(-\Delta) = -j(\Delta)$. So, we find that the magnetization current changes its parity from even to odd as a function of the anisotropy Δ :

$$j(\Delta) = j(-\Delta) \text{ for } N \text{ even,} \quad (13)$$

$$j(\Delta) = -j(-\Delta) \text{ for } N \text{ odd.} \quad (14)$$

We argue below, see sec.IV A, that at the free fermion point $\Delta = 0$ the spin current vanishes always, $j(\Delta = 0) = 0$. For odd N it follows from (14), while for even N it follows from a perturbative expansion. In addition, we make another suprising observation: due to (13), (14) the spin current alternates its sign with the system size since by increasing a system size by one unit $N \rightarrow N + 1$ the current dependence on Δ changes from even to odd or vice versa. For our case it amounts to

$$\text{sign } j(\Delta, N) = (-1)^N \text{ for } \Delta \leq -1, \quad (15)$$

because in the region $|\Delta| < 1$ the sign of $j(\Delta)$ can change depending on other parameters, see sec.IV B. The alternating current effect is a consequence of an existence of different LME symmetries for even and odd number of sites (10), (11) and as such is quite general. Indeed the derivation of (13), (14) does not depend neither on coupling Γ now on gradients κ and κ' as given by (4). Also, the phenomenon of alternating current does not rely on an integrability of the XXZ model. Indeed, an addition of a staggered anisotropy to the Hamiltonian (2) breaks integrability but the Eqs. (10), (11) remain valid. Moreover, even though our conclusion relied on the validity of Eqs. (13), (14), the alternating current was also observed in a recent study of a driven XXZ chain with an XY boundary setup different from (4), where one of two symmetries (Eq(13)) was absent [32]. We suggest that the magnetization current sign alternation resulting from applying transverse gradients can be a rather general phenomenon, which might be observable under appropriate experimental conditions in artificially assembled nanomagnets consisting of just a few atoms [41].

Another consequence of the current sign alteration is that the spin current *cannot be ballistic*. Indeed, for large N we should not be able to differentiate between even and odd N , so that the stationary magnetization current j must vanish in the thermodynamic limit:

$$j|_{N \rightarrow \infty} = 0 \text{ for all } \Delta, \Gamma. \quad (16)$$

This conclusion is a rather unexpected one, since in the XXZ integrable system, in the critical region $|\Delta| < 1$ one usually expects a ballistic current [25]. The magnetization current is also ballistic $|\Delta| < 1$ for reservoirs creating boundary gradients along the anisotropy axis [33].

Up to now we did not make use of the symmetry (12). In fact, it does not impose any further restrictions for spin current, while the energy current changes its sign under the symmetry (12) and must therefore vanish

$$J^Q = 0 \text{ for all } N, \Delta, \Gamma. \quad (17)$$

Consequently, the energy and spin currents decouple: the spin current can flow, while the energy current is totally suppressed (17). Vanishing of the energy current is a consequence of a symmetric choice $\kappa = \kappa'$ (7), and will be lifted if $\kappa \neq \kappa'$.

Remark. Note that if both boundary reservoirs were aligning boundary spins along the same direction in XY -plane, the magnetization current will be completely suppressed for all Δ , despite the existence of boundary gradients. E.g. for a choice of the set of Lindblad operators $V_1 = \alpha(\sigma_N^y + i\sigma_N^z)$, $V_2 = 0$, $W_1 = \beta(\sigma_N^y + i\sigma_N^z)$, $W_2 = 0$ in (3), which favours the alignment of the boundary spins along the X axis, the current suppression follows from a symmetry $\rho = \Sigma_x \rho \Sigma_x$ of the NESS, since $\text{Tr}(\hat{j}_{n,n+1}\rho) = -\text{Tr}(\hat{j}_{n,n+1}\Sigma_x \rho \Sigma_x)$. Thus, our choice (3) is the minimal nontrivial one, to observe a nonzero magnetization current.

In the following we check our findings and analyze the model further by computing exact NESS numerically for finite Γ and analytically for $\Gamma \rightarrow \infty$ using a perturbative expansion.

IV. STRONGLY DRIVEN XXZ CHAIN WITH XY TWIST: NUMERICAL AND ANALYTICAL STUDY.

We performed studies of the LME for small system sizes N in order to support our findings (13-17), which should be valid for all N, Δ and Γ . For simplicity we restrict here to the strongly driven case $\kappa = \kappa' = 1$. Being a problem of exponential growing complexity, the steady state for a finite Γ is a very complicated function of the parameters Γ, Δ , for any $N > 3$, and cannot be reported otherwise than in a graphical form. We solve the linear system of equations which determine the full steady state, numerically, making use of the global symmetry (12) which decrease the number of unknowns by roughly the factor of 1/2.

For $N = 3, 4$ the magnetization currents as function of the anisotropy, for various values of Γ are reported in Fig.2,3. As expected from (13),(14), the current is an odd (even) function of Δ for $N = 3(N = 4)$. Interestingly, as the coupling Γ increases, the current $j(\Gamma, \Delta)$ approaches quickly a limiting curve drawn by a dashed line. For the dashed line, which corresponds to the limit of infinitely strong couplings $\Gamma \rightarrow \infty$, a simple closed analytic form can be obtained. Here it is worthwhile to note that Lindblad Master equation dynamics (1) with large effective coupling to a reservoir $\Gamma \gg 1$ is realizable within the experimental protocol involving repeated periodic interactions of a boundary spin with a infinite bath system of identical ancilla [29].

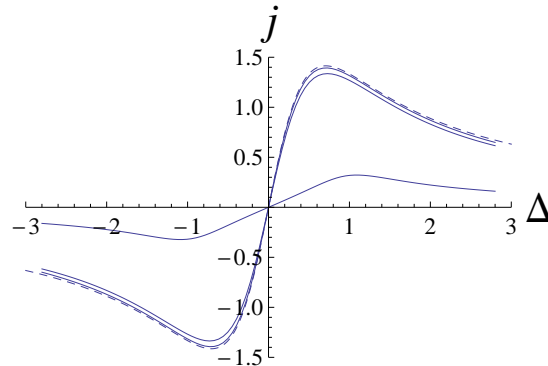


Figure 2: NESS spin current j vs Δ for $N = 3$ for different $\Gamma = 0.5, 5, 10$ (from bottom up). For finite Γ , the data are obtained numerically. Dashed line marks $j(\Delta)$ in the limit $\Gamma \rightarrow \infty$ from (29)

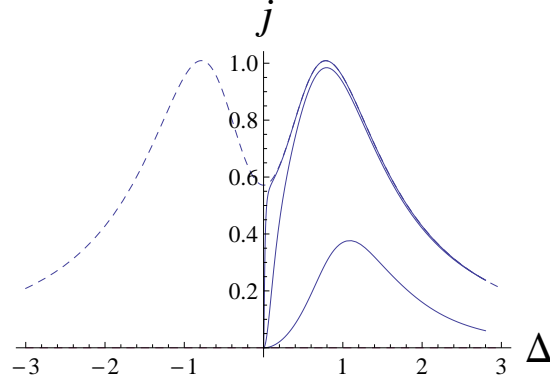


Figure 3: NESS spin current j vs Δ for $N = 4$ for different $\Gamma = 0.5, 5, 50$ (from bottom up). The $j(\Delta)$ curves continue symmetrically in the negative Δ region. For finite Γ the data are obtained numerically. Dashed line marks $j(\Delta)$ in the limit $\Gamma \rightarrow \infty$ from (30).

To investigate the strong coupling limit $\Gamma \gg 1$ we search for a stationary solution of the Lindblad equation in the form of a perturbative expansion

$$\rho_{NESS}(\Delta, \Gamma) = \sum_{k=0}^{\infty} \left(\frac{1}{2\Gamma} \right)^k \rho_k(\Delta). \quad (18)$$

As stressed in the end of Sec.II, the stationary solution $\rho_{NESS}(\Delta, \Gamma)$ is unique. The zeroth order term of the expansion (18) $\rho_0 = \lim_{\Gamma \rightarrow \infty} \lim_{t \rightarrow \infty} \rho(\Gamma, \Delta, t)$ satisfies $\mathcal{L}_{LR}[\rho^{str}] = 0$ (here and below we denote by $\mathcal{L}_{LR} = \mathcal{L}_L + \mathcal{L}_R$ the sum of the Lindblad actions in (1)). This enforces a factorized form

$$\rho_0 = \rho_L \otimes \left(\left(\frac{I}{2} \right)^{\otimes N-2} + M_0(\Delta) \right) \otimes \rho_R, \quad (19)$$

where ρ_L and ρ_R are one-site density matrices (5), (6) and M_0 is a matrix to be determined self-consistently later. We shall drop Δ - and N -dependence in ρ for brevity of notations. We separate the identity matrix $\left(\frac{I}{2} \right)^{\otimes N-2}$ for future convenience, so that $Tr(\rho_0) = 1$ and M_0 is traceless.

Inserting the (18) into (1), and comparing the orders of Γ^{-k} , we obtain recurrence relations

$$i[H, \rho_k] = \frac{1}{2} \mathcal{L}_{LR} \rho_{k+1}, \quad k = 0, 1, 2, \dots$$

A formal solution of the above is $\rho_{k+1} = -2\mathcal{L}_{LR}^{-1}(Q_{k+1})$ where $Q_{k+1} = -i[H, \rho_k]$. Note however that the operator \mathcal{L}_{LR} has a nonempty kernel subspace, and is not invertible on the elements from it. The kernel subspace $\ker(\mathcal{L}_{LR})$ consists of all matrices of type $\rho_L \otimes A \otimes \rho_R$ where A is an arbitrary $2^{N-2} \times 2^{N-2}$ matrix. Therefore a necessary and sufficient condition for ρ_{k+1} to exist is to require a null overlap

$$[H, \rho_k] \cap \ker(\mathcal{L}_{LR}) = \emptyset \quad (20)$$

We shall name (20) the secular conditions. For our choice of the Lindblad operator the secular condition are equivalent (see the Appendix) to the requirement of a null partial trace

$$Tr_{1,N}([H, \rho_k]) = 0, \quad k = 0, 1, 2, \dots \quad (21)$$

The remaining missing ingredient of the perturbation theory is obtained by noting that ρ_{k+1} is defined up to an arbitrary element from $\ker(\mathcal{L}_{LR})$, so we have

$$\rho_{k+1} = 2\mathcal{L}_{LR}^{-1}(i[H, \rho_k]) + \rho_L \otimes M_{k+1} \otimes \rho_R, \quad k = 0, 1, 2, \dots \quad (22)$$

Eqs (19), (22) and (21) define a perturbation theory for the Lindblad equation (1). At each order of the expansion, we must satisfy the secular conditions (20), before proceeding to the next order. Construction of the inverse \mathcal{L}_{LR}^{-1} is illustrated in the Appendix.

Applying the perturbation theory to our model, we find that the unknown apriori matrices $M_{2k}(\Delta)$, $M_{2k+1}(\Delta)$ are fully determined by secular conditions (21) for $2k, 2k+1$, as will be illustrated further on an example. For any $\Delta \neq 0$ the set of matrices $\{M_k(\Delta)\}$ is nontrivial. The case of zero anisotropy $\Delta = 0$ turns out to be special in that all $M_k \equiv 0$ and is discussed separately below.

A. "Free fermion" point $\Delta = 0$. Arbitrary N

For $\Delta = 0$ we can guess a zero approximation ρ_0 for arbitrary N :

$$\rho_0 = \rho_L \otimes \left(\frac{I}{2}\right)^{\otimes N-2} \otimes \rho_R. \quad (23)$$

Comparing with (19), we see that $M_0 = 0$. All further $M_k = 0$ and all secular conditions (21) are trivially satisfied at all orders. This fact was checked explicitly for small system sizes $N \leq 7$ and is conjectured to be true for arbitrary N . Consequently, the general recurrence (22) is reduced to $\rho_{k+1} = 2\mathcal{L}_{LR}^{-1}(i[H, \rho_k])$ for all k . At the first order of expansion, we obtain $i[H, \rho_0] = Q_1$, where

$$Q_1 = \frac{1}{2}\sigma^z \otimes \sigma^x \otimes \left(\frac{I}{2}\right)^{\otimes N-3} \otimes \rho_R \\ + \frac{1}{2}\rho_L \otimes \left(\frac{I}{2}\right)^{\otimes N-3} \otimes \sigma^y \otimes \sigma^z$$

which is easily inverted $\rho_1 = 2\mathcal{L}_{LR}^{-1}(Q_1) = -Q_1$ and so on. At all orders we notice $Tr(\hat{j}\rho_k) = 0$, which allows to conjecture that magnetization current vanishes for $\Delta = 0$ in spite of twisting boundary gradients,

$$j|_{\Delta=0} = 0 \quad \text{for all } N \geq 3 \text{ and all } \Gamma. \quad (24)$$

Note that for odd N , the statement (24) also follows from (14).

For $N = 3$, the perturbative expansion (18) can be summed up for all orders due to the fact that it closes at the third step: $\rho_3 = -4\rho_1$. Consequently, (18) is rewritten as

$$\rho_{N=3, \Delta=0}(\Gamma) = \rho_0 + \left(\frac{1}{2\Gamma}\rho_1 + \frac{1}{(2\Gamma)^2}\rho_2\right) \sum_{m=0}^{\infty} (-1)^m \frac{1}{\Gamma^{2m}}, \quad (25)$$

where $\rho_1 = -\frac{1}{2}(\sigma^z \otimes \sigma^x \otimes \rho_R + \rho_L \otimes \sigma^y \otimes \sigma^z)$ and $\rho_2 = \frac{1}{2}\sigma^y \otimes I \otimes \rho_R - \frac{1}{4}\sigma^x \otimes \sigma^z \otimes \sigma^z + \frac{1}{4}\sigma^z \otimes \sigma^z \otimes \sigma^y + \frac{1}{2}\rho_L \otimes I \otimes \sigma^y$. Summing up the series, we obtain

$$\rho_{N=3, \Delta=0}(\Gamma) = \rho_0 + \frac{\Gamma}{2} \frac{\rho_1}{\Gamma^2 + 1} + \frac{\rho_2}{4(\Gamma^2 + 1)}. \quad (26)$$

It can be verified straightforwardly that (26) is a stationary solution of LME (1) for $N = 3, \Delta = 0$. The sum (25) converges for $\Gamma > 1$. However, since the LME solution is unique, the expression (26) is valid for any Γ . In particular, for $\Gamma \rightarrow 0$ we obtain the weak coupling limit $\rho^{weak}(\Delta) = \lim_{\Gamma \rightarrow 0} \lim_{t \rightarrow \infty} \rho(\Gamma, \Delta, t)$, which is also unique by a continuity argument,

$$\rho_{N=3, \Delta=0}(\Gamma \rightarrow 0) = \rho^{weak}(\Delta)|_{\Delta=0} = \rho_0 + \frac{1}{4}\rho_2.$$

As expected, $\rho^{weak}(\Delta)|_{\Delta=0}$ commutes with the $XX0$ Hamiltonian, $[\rho_0 + \frac{1}{4}\rho_2, H_{XX0}] = 0$, and corresponds to zero currents, $j(\Gamma \rightarrow 0) = 0$.

From the above example we see that perturbative series (18) are useful even outside of their radius of convergence: if the perturbative series for ρ_{NESS} or for a particular observable $\langle \hat{f} \rangle = Tr(\hat{f}\rho_{NESS})$ can be summed, the result of the summation is valid for all Γ within its analyticity range, see also (35), (36).

B. Exact NESS for small system sizes and arbitrary Δ .

For any nonzero Δ , the matrices M_k from (22) are nontrivial and are determined from secular conditions (21). The secular conditions for two consecutive orders $2m, 2m+1$ reduce to a system of linear equations which determine M_{2m} and M_{2m+1} . In such a way, the perturbative series (18) is uniquely defined at all orders.

Below we present analytic results for M_0 and small N which define the NESS through Eq.(19). M_0 and M_1 are obtained by satisfying secular conditions (21) for $k = 0$ and $k = 1$. The procedure is illustrated below for the simplest case $N = 3, 4$. For $N = 3$, the most general form of the matrices $M_0(\Delta)$, $M_1(\Delta)$, by virtue of the symmetry (12), is $M_0 = b_0(\sigma^x - \sigma^y)$, $M_1 = b_1(\sigma^x - \sigma^y)$ where b_0, b_1 are unknown constants. The secular conditions (21) for $k = 0$ do not give any constraints on b_0 , while those for $k = 1$ give two nontrivial relations $b_1 = 0$ and $-\Delta + (1 + 2\Delta^2)b_0 = 0$ from which both M_0 and M_1 are

determined. For $N = 4$ each of the matrices $M_0(\Delta)$, $M_1(\Delta)$ compatible with the symmetry (12), contains 9 unknowns, all fixed by the secular conditions (21) for $k = 0, 1$, and so on. For $N = 3, 4$ we obtain

$$M_0(\Delta)|_{N=3} = \frac{\Delta}{1+2\Delta^2}(\sigma^x - \sigma^y) \quad (27)$$

$$\begin{aligned} s(\Delta)M_0(\Delta)|_{N=4} = & \left(-\frac{3\Delta^2}{2} + 2\Delta^4\right)\sigma^z \otimes \sigma^z + \left(\frac{1}{2} - 2\Delta^2\right)\sigma^x \otimes \sigma^y \\ & - 4\Delta^2\sigma^y \otimes \sigma^x + \left(\frac{1}{2} + 2\Delta^2\right)(\sigma^x \otimes I - I \otimes \sigma^y) \\ & + \frac{\Delta(4\Delta^2 + 5)}{2}(I \otimes \sigma^x - \sigma^y \otimes I) \\ & + 2\Delta(\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y). \end{aligned} \quad (28)$$

where $s(\Delta) = 7 + 6\Delta^2 + 8\Delta^4$. Matrices M_0 obey the symmetries (10)–(12). Steady state spin currents $j_0(\Delta) = \text{Tr}(\rho_0 \hat{j})$ in the limit $\Gamma \rightarrow \infty$ are

$$j_0(\Delta)|_{N=3} = \frac{4\Delta}{1+2\Delta^2}, \quad (29)$$

$$j_0(\Delta)|_{N=4} = \frac{4(1+4\Delta^2)}{7+6\Delta^2+8\Delta^4},$$

while the energy current vanishes $J^Q = 0$ as expected from (17). We note that the current for $N = 4$ is nonzero at $\Delta = 0$ which seems to contradict our prediction (24). The contradiction arises because the limits $\Gamma^{-1} \rightarrow 0$ and $\Delta \rightarrow 0$ do not commute. Indeed, we find that even orders of the perturbative expansion ρ_{2m} contain terms which become singular for $\Delta \rightarrow 0$. The leading singular term at the order $2m$ of the expansion is $\text{Tr}(\varepsilon^{2m} \rho_{2m} \hat{j}) \sim (\varepsilon/\Delta)^{2m}$, where $\varepsilon = \Gamma^{-1}$. Since the singularities become stronger with m , for any given infinitesimal ε there will be a region $|\Delta| \lesssim \varepsilon$ where the expansion (18) will diverge. The origin behind the non-commuting limits is the existence of an extra symmetry (13) of the NESS at the point $\Delta = 0$. Similar situation is encountered e.g. in Kubo linear response theory, where a nonergodicity at zero point in the momentum space $\vec{q} = 0$ is inflicted by an extra conservation law at this point [42]. Reversing order of the limits $\varepsilon \rightarrow 0$ and $\Delta \rightarrow 0$ does yield $j(0) = 0$ for any Γ , so that the current expression for $N = 4, \Gamma \rightarrow \infty$ has to be modified as

$$j_0(\Delta)|_{N=4} = \begin{cases} \frac{4(1+4\Delta^2)}{7+6\Delta^2+8\Delta^4}, & \text{if } \Delta \neq 0 \\ 0, & \text{if } \Delta = 0, \end{cases} \quad (30)$$

while $M_0|_{N=4, \Delta=0} = 0$ as given by (23). For any finite Γ the dependence $j(\Delta)$ is smooth and does not contain any singularities, see Fig.3. We find the noncommutativity of the limits $\varepsilon \rightarrow 0, \Delta \rightarrow 0$ for all even system sizes $N = 2m \geq 4$, see also (32). Since for large N one should not be able to differentiate between even and odd system sizes, we expect that the $j_0(\Delta)$ for $\Delta \ll 1$ vanishes in the thermodynamic limit: $\lim_{m \rightarrow \infty} j_0(\Delta \ll 1)|_{N=2m} = 0$.

The number of equations to solve (21) at each order of perturbation theory grows exponentially with the system size N , although the symmetry (12) restricts the number of independent variables. The matrices M_0 for $N \geq 5$ are given by lengthy expressions and are not reproduced here. Instead, we report NESS currents at $\Gamma \rightarrow \infty$, given for $N = 5, 6$ by

$$j_0|_{N=5} = \frac{4\Delta^3(-3+12\Delta^2+2^3\Delta^4+2^5\Delta^6)}{1-8\Delta^2-6\Delta^4+120\Delta^6+96\Delta^8-2^6\Delta^4+2^7\Delta^{12}} \quad (31)$$

$$j_0|_{N=6} = \begin{cases} \frac{4(12-3\Delta^2-220\Delta^4+48\Delta^6+2^6 19\Delta^8-2^8 5\Delta^{10}+2^{10}\Delta^{12})}{149-604\Delta^2+316\Delta^4-912\Delta^6+4992\Delta^8-5760\Delta^{10}+15360\Delta^{12}-2^{14}\Delta^{14}+2^{13}\Delta^{16}}, & \text{if } \Delta \neq 0. \\ 0, & \text{if } \Delta = 0 \end{cases} \quad (32)$$

The NESS currents j_0 for odd/even number of sites N are even/odd functions of Δ respectively, as expected from (13), (14) and depend rather nontrivially on the anisotropy Δ , see Fig.4. In particular for $N \geq 5$ one observes sharp peaks in the $j_0(\Delta)$ resembling resonances at which the current changes its sign and amplitude. One gets an insight by comparing average magnetization profiles $\langle \sigma_n^x \rangle$ and $\langle \sigma_n^y \rangle$ for values of Δ inside and outside the resonance region. The average magnetization at boundaries $n = 1, N$ is fixed by the Lindblad reservoirs $\langle \sigma_1^x \rangle \rightarrow 0, \langle \sigma_N^x \rangle \rightarrow 1$ and $\langle \sigma_1^y \rangle \rightarrow 1, \langle \sigma_N^y \rangle \rightarrow 0$ so that the lowest interpolating Fourier mode has the wavelength $\Lambda/a = 4(N-1)$ where a is the lattice constant. In the

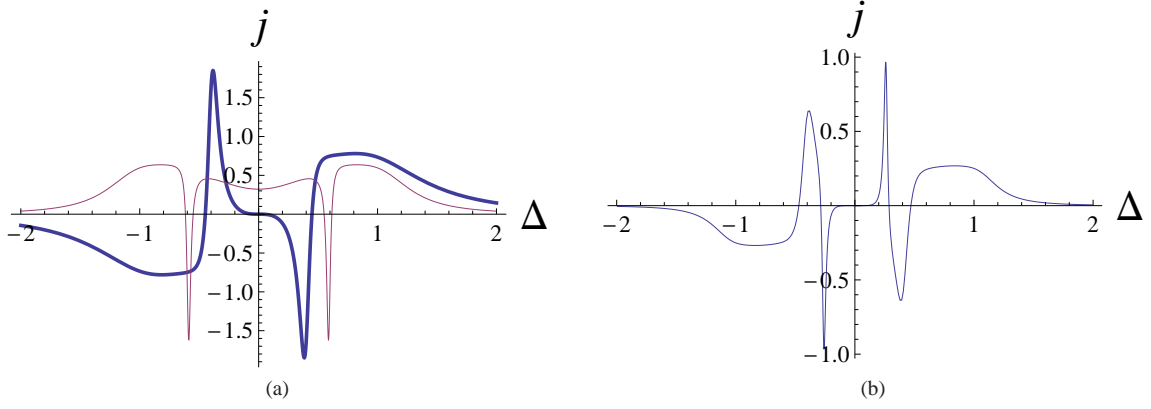


Figure 4: NESS spin current vs Δ for $N = 5, 6$ from Eqs(31),(32) given by thick and thin line respectively (Panel (a)), and for $N = 7$ (Panel (b)), all for $\Gamma \rightarrow \infty$.

resonance region we observe that $\Lambda' = \Lambda/3$ becomes the dominant Fourier mode, see Fig.5(b). One can interpret this frequency tripling as a triple increase of a twisting angle along the chain. Note, see Fig.5(a,b) that there is a $\pi/2$ phase shift between the dominant harmonics describing $\langle \sigma_n^x \rangle$ and $\langle \sigma_n^y \rangle$ profiles. In particular, in the resonance region, Fig.5(b), $\langle \sigma_n^x \rangle \approx -\sin \phi(n-1)$, $\langle \sigma_n^y \rangle \approx -\cos \phi(n-1)$, where $\phi = 3\pi/10$. In the meanfield approximation the respective magnetization current is $j_{MF} = 2(\langle \sigma_n^x \rangle \langle \sigma_{n+1}^y \rangle - \langle \sigma_n^y \rangle \langle \sigma_{n+1}^x \rangle) = -2 \sin \phi \approx -1.61803$, which is amazingly close to the exact current value at the resonance $j_{exact} = -1.61857\dots$ With the increase of system size, one might expect an appearance of further resonances at wavelengths $\Lambda' = \Lambda/5, \Lambda/7, \dots$ corresponding to larger twisting angles $\frac{5\pi}{2}, \frac{7\pi}{2}$ etc.. Why higher Fourier harmonics become stable under a strong Lindblad action for narrow lacunae of Δ , and how to predict a location of the peaks remains an intriguing open question. As coupling Γ decreases, the resonances become smoother and then ultimately disappear, but their precursors can still be seen for Γ of the order of the Heisenberg exchange interaction and larger (data not shown).

Examining analytic expressions for the current for small N , we can extrapolate its behaviour for $\Delta \gg 1$ and $0 < \Delta \ll 1$ and finite N as

$$j|_{\Gamma \rightarrow \infty, \Delta \gg 1} = O\left(\frac{1}{\Delta^{N-2}}\right) \quad (33)$$

$$j|_{\Gamma \rightarrow \infty, 0 < \Delta \ll 1} = O((-1)^{\frac{N+1}{2}} \Delta^{N-2}) \text{ for odd } N \quad (34)$$

The Eq.(33) implies an exponential decay of the current with the system size N for fixed large Δ , $j|_{\Gamma \rightarrow \infty, \Delta \gg 1} = O(e^{-\alpha N})$ with $\alpha = \ln \Delta$, and is generically expected from an insulating nature of the XXZ chain in the $\Delta \gg 1$ Ising limit. Similar behaviour of the NESS current at large Δ was found for driven chain with Z -polarized boundary baths [23]. On the other hand, Eq.(34) implies a complete flattening of the current near $\Delta = 0$ in the thermodynamic limit $N \rightarrow \infty$, which is a new unusual feature. The flattening is already well seen in Fig.4 for $N = 5, 7$. Note that the results (33), (34) would be very difficult to obtain by studying perturbative series in Δ or in $1/\Delta$: in both cases $\Delta \ll 1$ and $1/\Delta \ll 1$ the first nonzero term will appear only in the order $N - 2$ of the expansion!

C. Generic Γ and arbitrary driving κ

The analytic results derived in previous subsection concerned the limits of $\Gamma \rightarrow \infty$ and strong driving $\kappa = \kappa' = 1$. What can we say about generic Γ and κ ? Examining perturbative series (18) for various N and Δ we notice that nonzero contributions to NESS current appear only at even orders of the expansion Γ^{-2n} . If the NESS current $j(\Gamma, \Delta)$ is an analytic function of Γ , then we can expect the current to be an even function of Γ . In particular, for small $\Gamma \ll 1$ one expects the the current to behave as $j(\Gamma \ll 1) = O(\Gamma^2)$, since the current must vanish at zero coupling $\Gamma = 0$. Our expectations are confirmed by exact expressions for the current as function of Γ , obtained for special case of isotropic XXX model, for which one can sum up the series (18)

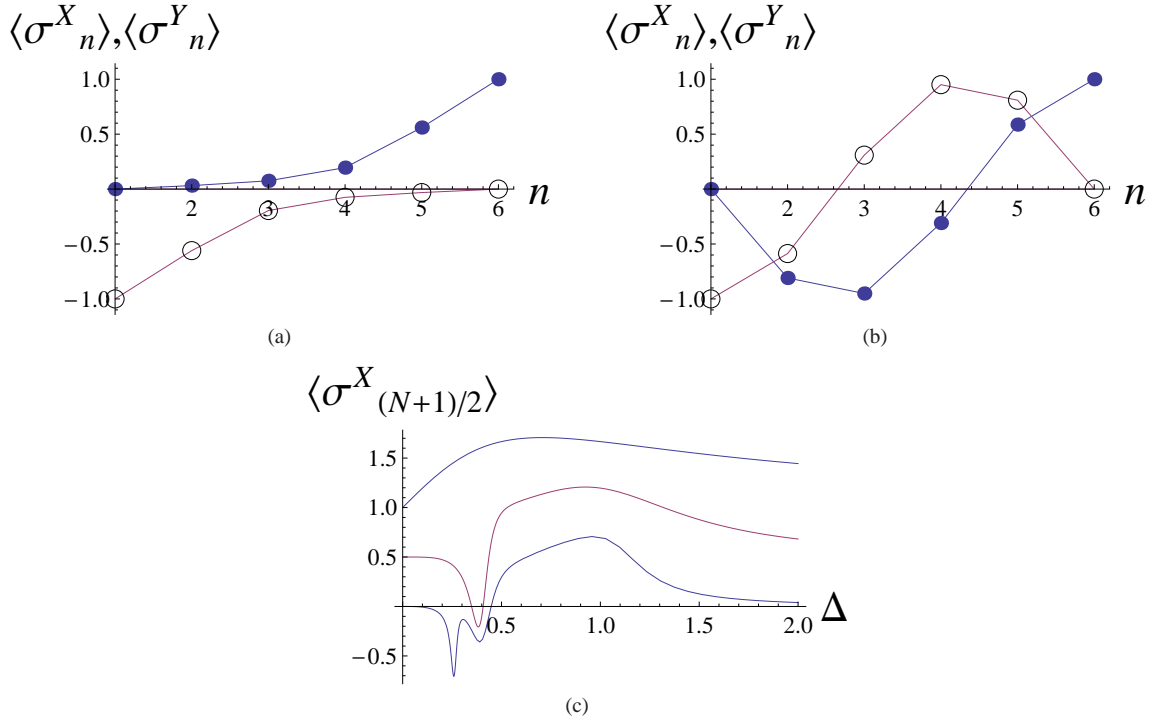


Figure 5: Typical $\langle \sigma_n^X \rangle$ -profiles (filled circles) and $\langle \sigma_n^Y \rangle$ - profiles (open circles) outside the resonance (Panel (a), $\Delta = 1.8$, $N = 6$) and inside the resonance (Panel (b), $\Delta = 0.5875$, $N = 6$). Frequency tripling is clearly observed in the Panel (b). Panel (c) shows dependence of the average spin projection $\langle \sigma_n^X \rangle$ in the middle of the chain $n = (N + 1)/2$ on Δ , for $N = 3, 5, 7$ (from up to down). The curves for $N = 3, 5$ are shifted vertically by 1 and 0.5 for a better view.

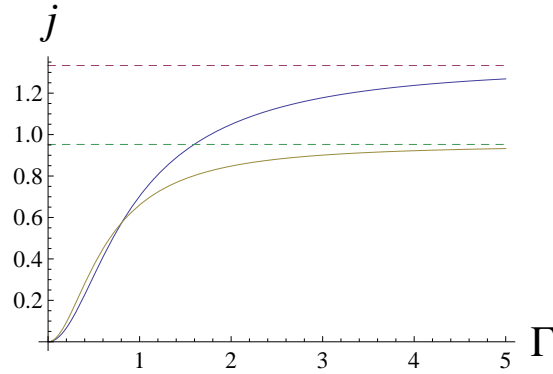


Figure 6: NESS spin current j vs Γ for $N = 3, 4$ (thick and thin line, respectively) for isotropic Heisenberg model $\Delta = 1$, given by (35),(36). Dashed lines indicate limiting values of the current for $\Gamma \rightarrow \infty$.

and obtain

$$j(\Gamma)|_{N=3, \Delta=1} = \frac{16\Gamma^2(3 + \Gamma^2)}{27 + 52\Gamma^2 + 12\Gamma^4} \quad (35)$$

$$j(\Gamma)|_{N=4, \Delta=1} = \frac{8\Gamma^2(27 + 46\Gamma^2 + 10\Gamma^4)}{3(27 + 136\Gamma^2 + 144\Gamma^4 + 28\Gamma^6)}. \quad (36)$$

Indeed the above expressions contain only even Γ orders and $j(\Gamma \ll 1) = O(\Gamma^2)$, see also Fig.6. Numerically, we observe $j(\Gamma \ll 1) = O(\Gamma^2)$ for other Δ values as well. Generically, we find the current amplitude to be a smooth function of Γ for a fixed value of the anisotropy Δ . For large couplings, it asymptotically approaches its limiting value $j_0(\Delta)$ at $\Gamma \rightarrow \infty$.

Interestingly, the magnetization current is not suppressed for large couplings Γ , as it happens for Z -polarized Lindblad baths [23],[21] due to a quantum Zeno effect. Indeed for our choice of the bath (4) and large coupling, the quantum Zeno effect "freezes" σ^y and σ^x spin components at the boundaries, while hoppings in the σ^z component, contributing to the magnetization current, may still occur.

Finally, we investigated the current dependence on the boundary gradient κ , which can be manipulated by a choice of the amplitudes of the Lindblad operators in (4). By Fourier law, we may expect the current to be proportional to κ , for small gradients κ , and introduce a conductivity tensor $\chi_{\alpha\beta}$ as a proportionality coefficient between the current $j \equiv j^z$ and the infinitesimal boundary gradient $\langle \delta\sigma^\beta \rangle / N$ in the spin component β , $j^\alpha = \chi_{\alpha\beta} \langle \delta\sigma^\beta \rangle$.

With our perturbative method we construct iteratively an expansion (18) for a NESS for an arbitrary amplitude of the boundary gradient κ from (4). We observe that the current is a rather nontrivial function of κ , and in particular that it vanishes as κ^2 or faster for small κ . Consequently, $\lim_{\kappa \rightarrow 0} j(\kappa)/\kappa = 0$ so that the off-diagonal elements of the conductivity tensor χ_{zx}, χ_{zy} vanish. Note that diagonal elements of the conductivity tensor are nonzero [21]. Numerically, we find that the leading order of expansion of $j(\kappa)/\kappa$ for small κ depends on N and Δ in a nontrivial way. More details will be given elsewhere.

Finally, note that our model shows a behaviour fundamentally different from that with usually considered boundary driving along the Z -axis, referred to below as a "scalar" setup. Just to mention two crucial differences: the scalar setup leads to a ballistic current in the critical gapless region of the XXZ model $\Delta < 1$ [23], while in the XY -driven chain the alternating signs phenomenon (10-15) rules out the possibility of a ballistic current. In the "scalar" setup the current at strong coupling is suppressed due to a Zeno effect, $j_{\Gamma \rightarrow \infty} \rightarrow 0$, while in the XY -driven chain of finite size the spin current in the limit $\Gamma \rightarrow \infty$ remains finite.

V. CONCLUSIONS

We considered an open XXZ spin 1/2 chain with a twisting in XY - plane, imposed by boundary gradients. Symmetries of the Lindblad equation were found which impose drastic restrictions on the steady state reduced density matrix, and various observables. For our boundary setup the NESS energy current vanishes, and the magnetization current alternates its sign with system size, which rules out the possibility of a ballistic current. We argued that the current sign alternation under trasverse gradients is a robust phenomenon in quantum transport which does not rely on integrability. The qualitative difference in quantum transport in even- and odd-sized systems adds to a list of similar phenomena observed in quantum diffusion in periodic chains [43]. To support our findings, we constructed a nonequilibrium steady state solution of the Lindblad Master equation in the form of a perturbative expansion in orders of Γ^{-1} , and calculated the zeroth-order term analytically for small system sizes, by solving a set of linear equations which guarantee the self-consistency of the expansion, the secular conditions (21). We find that, for $\Delta \neq 0$, the current remains finite even in the limit of infinitely strong effective coupling.

Further on, for large couplings we find a nontrivial dependence of the magnetization current on the anisotropy $j(\Delta)$, and observed sharp peaks in $j(\Delta)$ inside the critical gapless region of the XXZ model $\Delta < 1$, the origin of which was attributed to an appearance of higher Fourier harmonics in magnetization profiles. We observe an anomalous flattening of the current near $\Delta = 0$ point for odd N and noncommutativity of limits $\Delta \rightarrow 0$ and $\Gamma^{-1} \rightarrow 0$ for even N . For weak driving $\kappa \ll 1$ and small system sizes, we find $\lim_{\kappa \rightarrow 0} j(\kappa)/\kappa = 0$ for all values of Δ , signalinging subdiffusive current in the direction trasverse to XY boundary gradients.

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Appendix A: Inverse of the Lindblad dissipators and secular conditions.

\mathcal{L}_L and \mathcal{L}_R are linear super-operators acting on a matrix ρ as defined by (3). In our case, each super-operator act locally on a single qubit only. We find the eigen-basis $\{\phi_R^\alpha\}_{\alpha=1}^4$ of $\frac{1}{2}\mathcal{L}_R\phi_R^\alpha = \lambda_\alpha\phi_R^\alpha$ in the form $\phi_R = \{\rho_R, \sigma^x, \sigma^y, \sigma^z\}$, where $\sigma^x, \sigma^y, \sigma^z$ are Pauli matrices and $\rho_R = \frac{1}{2}(I + \kappa\sigma^x)$. The respective eigenvalues are $\{\lambda_\alpha\} = \{0, 2, 1, 1\}$. Analogously, the eigen-basis and eigenvalues of the eigenproblem $\frac{1}{2}\mathcal{L}_L\phi_L^\beta = \mu_\beta\phi_L^\beta$ are $\phi_L = \{\rho_L, \sigma^x, \sigma^y, \sigma^z\}$ and $\{\mu_\beta\} = \{0, 1, 2, 1\}$, where $\rho_L = \frac{1}{2}(I - \kappa\sigma^y)$. Since the bases ϕ_R and ϕ_L are complete, any matrix F acting in the appropriate Hilbert space can be expanded as

$$F = \sum_{\alpha=1}^4 \sum_{\beta=1}^4 \phi_L^\beta \otimes F_{\beta\alpha} \otimes \phi_R^\alpha \quad (\text{A1})$$

in the unique way. Indeed, let us introduce complementary bases $\psi_R = \{I, \frac{1}{2}(\sigma^x - \kappa I), \frac{1}{2}\sigma^y, \frac{1}{2}\sigma^z\}$ and $\psi_L = \{I, \frac{1}{2}\sigma^x, \frac{1}{2}(\sigma^y + \kappa I), \frac{1}{2}\sigma^z\}$, trace-orthonormal to the ϕ_R, ϕ_L respectively, $\text{Tr}(\psi_R^\gamma \phi_R^\alpha) = \delta_{\alpha\gamma}$, $\text{Tr}(\psi_L^\gamma \phi_L^\beta) = \delta_{\beta\gamma}$. Then, the coefficients of the

expansion (A1) are given by $F_{\beta\alpha} = Tr_{1,N}((\psi_L^\beta \otimes I^{\otimes N-1})F(I^{\otimes N-1} \otimes \psi_R^\alpha))$. On the other hand, in terms of the expansion (A1) the superoperator inverse $(\mathcal{L}_L + \mathcal{L}_R)^{-1}$ is simply

$$-2(\mathcal{L}_L + \mathcal{L}_R)^{-1}F = \sum_{\alpha,\beta} \frac{1}{\lambda_\alpha + \mu_\beta} \phi_L^\beta \otimes F_{\beta\alpha} \otimes \phi_R^\alpha. \quad (\text{A2})$$

Note however that the above sum contains a singular term with $\alpha = \beta = 1$, because $\lambda_1 + \mu_1 = 0$. To eliminate the singularity, one must require $F_{11} = Tr_{1,N}F = 0$, which produces the secular conditions (21).

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